# The Development of Everywhere Continuous, Nowhere Differentiable Functions

Halley Winkler

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Approved by:

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Rene Salinas, Ph.D., Thesis Director

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Scott T. Marshall, Ph.D., Second Reader

---

Quinn Morris, Ph.D., Second Reader

---

William J. Cook, Ph.D., Honors Director, Department of Mathematical Sciences

---

Jefford Vahlbusch, Ph.D., Dean, The Honors College

# ABSTRACT

While researching everywhere continuous, nowhere differentiable functions, one would find a variety of papers and different strategies proving the majority of continuous functions to be nowhere differentiable. This is a sharp contrast to the beliefs of late eighteenth and early nineteenth century mathematicians. It was not only falsely believed that all continuous functions are differentiable except for at some isolated points, but mathematicians that sought to solidify analysis and the definition of continuity faced strong opposition from their peers.

Despite this opposition, many scholars became invested in formalizing mathematical logic and expanding on the ideas of the pioneers of pathological functions; like Weierstrass and Bolzano [1]. Functions like the Bolzano function and Weierstrass function, whose continuity and nowhere differentiability proofs are included in this paper, inspired the discovery of series and fractals with these same properties. This paper serves as a timeline of some of the most significant discoveries of everywhere continuous, nowhere differentiable functions, from the first publication of one in the late nineteenth century, to modern examples like Liu Wen's function in 2002.

# 1. INTRODUCTION

For novice calculus students, continuity is one of the concepts introduced earliest, and built upon for the longest. The theorem which states that if a function $f$ is differentiable at a point $a$, then $f$ is continuous at $a$, which can then be extended to state that a function $f$ that is differentiable on an interval $[a, b]$, is also continuous on the interval $[a, b]$ may come to an advanced student through intuition. A student might then instinctually believe the converse is also true, but a function being continuous at a point does not automatically imply it is also differentiable at that point.

One of the simplest examples that the converse is not true is the absolute value function $f = |x|$ at the point $x = 0$. Since $\lim_{x\to 0} |x| = 0$ , we know $f$ is continuous, but the $\lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$, is $-1$ from the left, and 1 from the right, so we know $f$ is not differentiable when $x = 0$. Finding out a continuous function is not necessarily differentiable at every point defies intuition, and naturally leads to the question of whether or not a function exists which is always continuous, but never differentiable. This was also the natural progression of learning in the 19th century, when Karl Weierstrass became the first mathematician to publicly present a piece proving the existence of everywhere continuous, nowhere differentiable functions [1].

These functions are significant for a multitude of reasons. The discovery and interest in curves with these properties marked an ideological shift in the math community at the time. Innovations in math prior to this had been rooted in practicality or the need to define scientific processes, and early work on these functions were a major departure from this pragmatic attitude towards math. Pathological discoveries helped emphasize the need for rigorous and precise definitions in math and inspired the development of mathematical logic, similar to the logic I'll be utilizing in some of my proofs.

Unbeknownst to the academics who introduced continuous, nowhere differentiable functions, their functions are some of the earliest examples of fractal curves. Fractals are where these functions find most of their real-world applicability. In the following section, I'll delve into the earliest example of the everywhere continuous, nowhere differentiable functions.

# 2. THE WEIERSTRASS FUNCTION

> "The more I meditate on the principles of the theory of functions—and I do this unremittingly the stronger becomes my conviction that the foundations upon which these must be built are the truths of Algebra." – Karl Weierstrass

Weierstrass was invested in the solidity of calculus. More specifically, he looked to solidify the definitions for and distinguish between continuity and uniform continuity, which had been loosely defined already by Augustin-Louis Cauchy. He first presented his pathological function to the Berlin Academy in 1872 and was met with strong opposition [4]. Prior to this, mathematical discoveries had been mainly practical. New functions came about through applications and necessity, and mathematicians at the time did not appreciate a departure from this practicality.

Even his close friendships with other academics were damaged due to their opposition to Weierstrass's always continuous, never differentiable function. Many felt that Weierstrass's rigorous approach was inspired solely by wanting to contradict and disprove the mathematicians before him.

This in fairness, was not untrue. Weierstrass's function is considered a pathological example, which means it is an example whose purpose is to defy intuition and universally assumed-true properties. This same rigorous approach, however, would inspire the development of real analysis, as it swayed other mathematicians to look for ugly, contrary properties of functions, and led to the discovery of infinitely complicated topics like fractals [3].

The following is a proof that the function presented by Weierstrass in 1872 is continuous at every real-numbered point, but differentiable nowhere [2]. This function can be defined as

$$f(x) = \sum_{n=0}^{\infty} a^n cos(b^n x\pi) \tag{2.1}$$

where $a$ is a real number such that $0 < a < 1$, and where $b$ is an odd, positive integer such that $\frac{\pi}{ab-1} < \frac{2}{3}$ and $ab > 1$.

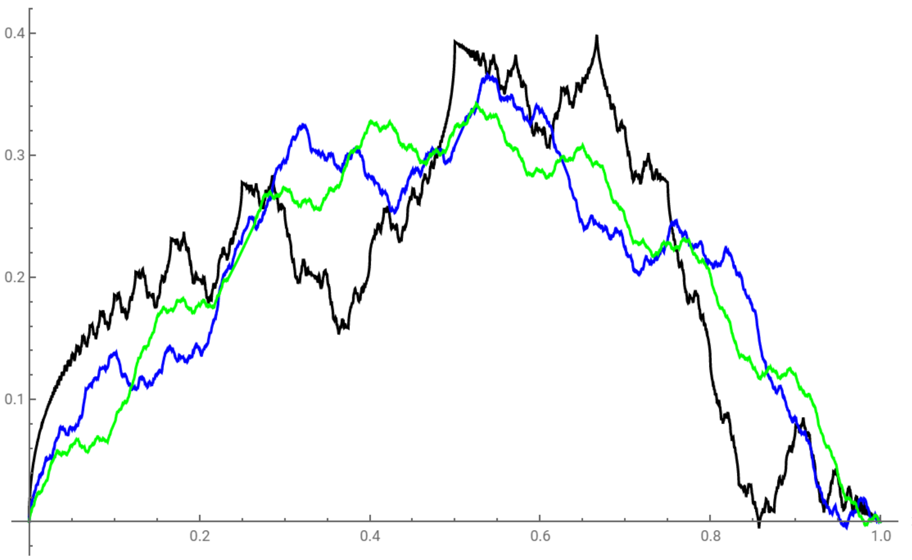


Figure 2.1: Above is a figure made in Mathematica which displays the second, third, and fourth iterations of the Weierstrass function, which are represented by black, blue, and green respectively [12].)

My proof will be organized into two sections. Firstly, I'll prove that $f(x)$ is continuous for any real number $x$, and then I'll prove that $f(x)$ is not differentiable at any point. The theorems and inequalities I chose were based on Johan Thim's proof for initial guidance, specifically for the differentiability section, however the way I achieved my results differs in that it is more dependent on arithmetic than using corollaries and Cauchy theorems [6].

**Theorem 2.0.1.** *Equation 2.1; $f(x) = \sum_{n=0}^{\infty} a^n cos(b^n x\pi)$ where $a$ is a real number such that $0 < a < 1$, and where $b$ is an odd, positive integer such that $\frac{\pi}{ab-1} < \frac{2}{3}$ and $ab > 1$. is continuous everywhere, but differentiable nowhere.*

*Proof.* Continuity is fairly obvious in this case if one considers Weierstrass's M Test. The M test theorem states

**Theorem 2.0.2.** *If $f_1, f_2, \cdots X \rightarrow \mathbb{R}$ is a sequence of functions from the set $X$ in the real number system, and there exists constants $M^k$, such that $|f_k(x)| \leq M_k$ is true for any $x \in X$ and $k \geq 1$, and $\sum_{k=1}^{\infty} M^k < \infty$, then it is also true that the series $\sum_{k=1}^{\infty} f_k(x)$ uniformly converges on $X$.*

Keeping this theorem in mind, consider that for any real number $x$, $cos(b^n \pi x)$ will be between $-1$ and $1$, and so

$$a^n cos(b^n \pi x)| \leq a^n. \tag{2.2}$$

Next, consider that $\sum_{n=0}^{\infty} a^n$ where $0 < a < 1$ is a geometric series which converges at every point, so we can say by Theorem 2.0.2 that

$$f(x) = \sum_{n=0}^{\infty} a^n cos(b^n \pi x) \tag{2.3}$$

is convergent and continuous or any real number $x$.

Now that we've established that $f(x)$ is a continuous function, we must also prove it is not differentiable for any real number $x$.

Let there exist a natural number $k$ and let there also exist an integer $\beta_k$ such that $\frac{1}{2} \leq \beta_k - b^k x_0 < \frac{3}{2}$, where $x_0$ is some fixed value which satisfies the inequality. The inequality can also be expressed as

$$x_0 > \frac{\beta_k}{b^k} - \frac{3}{2b^k} \text{and} x_0 \leq \frac{\beta_k}{b^k} - \frac{1}{2b^k} \tag{2.4}$$

Then consider that since $b > 1$, $\lim_{k\to\infty} \frac{\beta_k}{b^k}$ is approaching $x_0$. Next, assume $f(x)$ can be differentiated at this point $x_0$, such that

$$\lim_{k\to\infty} \frac{f(\frac{\beta_k}{b^k}) - f(x_0)}{\frac{\beta_k}{b^k} - x_0} = f'(x_0). \tag{2.5}$$

To prove $f$'$(x_0)$ doesn't exist, and consequentially, that $f(x)$ is not differentiable for any real number $x$, I'll be showing that $\lim_{k\to\infty} \frac{f(\frac{\beta_k}{b^k}) - f(x_0)}{\frac{\beta_k}{b^k} - x_0}(-1)^{\beta_k}$ approaches $\infty$, therefore contradicting the statement we're considering.

First, consider that

$$\frac{f(\frac{\beta_k}{b^k}) - f(x_0)}{\frac{\beta_k}{b^k} - x_0}(-1)^{\beta_k} = (-1)^{\beta_k} \frac{\sum_{n=0}^{\infty} a^n cos(b^n \frac{\beta_k}{b^k}\pi) - \sum_{n=0}^{\infty} a^n cos(b^n x_0 \pi)}{\frac{\beta_k}{b^k} - x_0} \tag{2.6}$$

$$= \sum_{n=0}^{\infty} (-1)^{\beta_k} \frac{a^n cos(b^n \frac{\beta_k}{b^k}\pi) - a^n cos(b^n x_0 \pi)}{\frac{\beta_k}{b^k} - x_0}$$

$$= \sum_{n=0}^{\infty} (-1)^{\beta_k} (a^n) \frac{cos(b^n \frac{\beta_k}{b^k}\pi) - a^n cos(b^n x_0 \pi)}{\frac{\beta_k}{b^k} - x_0}$$

$$= \sum_{n=m}^{\infty}(-1)^{\beta_k}(a^n)\frac{cos(b^n\frac{\beta_k}{b^k}\pi) - a^n cos(b^n x_0\pi)}{\frac{\beta_k}{b^k} - x_0} + \sum_{n=0}^{m-1}(-1)^{\beta_k}(a^n)\frac{cos(b^n\frac{\beta_k}{b^k}\pi) - a^n cos(b^n x_0\pi)}{\frac{\beta_k}{b^k} - x_0} \tag{2.7}$$

For clarity, let the terms in equation 2.7 be defined to be$A_k$and$B_k$ respectively. Recall we want to show that
$\lim_{k\to\infty} \frac{f(\frac{\beta_k}{b^k})-f(x_0)}{\frac{\beta_k}{b^k}-x_0}(-1)^{\beta_k} = \infty$. So, if the following inequalities hold

$$(ab)^k\frac{\pi}{ab-1} \geq |A_k| \tag{2.8}$$

and

$$(ab)^k\frac{2}{3} \leq |B_k| \tag{2.9}$$

then we'll be able to show

$$\sum_{n=0}^{k-1}(-1)^{\beta_k}(a^n)\frac{cos(b^n\frac{\beta_k}{b^k}\pi) - a^n cos(b^n x_0\pi)}{\frac{\beta_k}{b^k} - x_0} \geq B_k - |A_k| \tag{2.10}$$

and

$$B_k - |A_k| \geq (ab)^k\frac{\pi}{ab-1} \tag{2.11}$$

which may not seem useful now, but will help us to complete our contradiction later and is important to keep in mind. To verify these inequalities, we'll first prove that

$$(ab)^k\frac{\pi}{ab-1} \geq |A_k| \tag{2.12}$$

To begin, consider that

$$A_k \cdot \frac{b^n\pi}{b^n\pi} = \sum_{n=0}^{k-1}(-1)^{\beta_k}(a^n)(b^n\pi)\frac{cos(b^n\frac{\beta_k}{b^k}\pi) - a^n cos(b^n x_0\pi)}{(\frac{\beta_k}{b^k} - x_0)\cdot(b^n\pi)}$$

$$= \sum_{n=0}^{k-1}(-1)^{\beta_k}(a^n)(b^n\pi)(-sin(c_{n,k})) \tag{2.13}$$

where by the Mean Value Theorem, we can assume $c_{n,m}$ exists within the interval $[a, b]$. Then,

$$|A_k| = |\sum_{n=0}^{k-1}(a^n)(b^n\pi)(-sin(c_{n,k}))| \leq \sum_{n=0}^{k-1}(a^n)(b^n\pi) = \pi\frac{(ab)^k - 1}{ab - 1}, \tag{2.14}$$

and

$$\pi\frac{(ab)^k - 1}{ab - 1} < (ab)^k\frac{\pi}{ab - 1} \tag{2.15}$$

Therefore,

$$(ab)^k\frac{\pi}{ab - 1} \geq |A_k|. \tag{2.16}$$

Now that we've verified equation 2.8, we'll move on and verify equation 2.9 To begin, recall the condition that $b$ is an odd, positive integer, so when $\beta_k$ is odd, we have that $cos(b^{n-k}\beta_k\pi) = -1$, and when $\beta_k$ is even, $cos(b^{n-k}\beta_k\pi) = 1$. This allows us to define a very useful equivalency;

$$cos(b^{n-k}\beta_k\pi)(-1)^{\beta_k} = (-1)^{\beta_k}(1)^{\beta_k} = +1 \tag{2.17}$$

Next, consider that

$$cos(b^n x_0\pi)(-1)^{\beta_k} = cos(b^{n-k}b^k x_0\pi)(-1)^{\beta_k}$$

$$= cos(b^{n-k}\beta_k\pi + b^{n-k}(b^k x_0 - \beta_k)\pi)(-1)^{\beta_k}$$

$$= (-1)^{\beta_k}(cos(b^{n-k}\beta_k\pi)cos(b^{n-k}(b^k x_0 - \beta_k)\pi)) - sin(b^{n-k}\beta_k\pi)sin((b^{n-k}\beta_k\pi)sin((b^{n-k}(b^k x_0 - \beta_k)\pi).$$

$$= (-1)^{\beta_k}(cos(b^{n-k}\beta_k\pi)cos(b^{n-k}(b^k x_0 - \beta_k)\pi)$$

$$= cos(b^{n-k}(b^k x_0 - \beta_k)\pi)$$

So

$$cos(b^n x_0\pi)(-1)^{\beta_k} = cos(b^{n-k}(b^k x_0 - \beta_k)\pi) \tag{2.18}$$

Next, we can use what we just calculated to show that

$$B_k = \sum_{n=k}^{\infty} a^n\frac{1 - cos((b^k x_0 - \beta_k)\pi)}{\frac{\beta_k}{b^k} - x_0}$$

$$= a^k\frac{1 - cos((b^k x_0 - \beta_k)\pi)}{\frac{\beta_k}{b^k} - x_0} + \sum_{n=k+1}^{\infty} a^n\frac{1 - cos(b^{n-k}(b^k x_0 - \beta_k)\pi)}{\frac{\beta_k}{b^k} - x_0} \tag{2.19}$$

Finally , consider that $\beta_k - b^k x_0 \geq \frac{1}{2}$, and $\beta_k - b^k x_0 < \frac{3}{2}$, so $\frac{\beta_k}{b^k} - x_0$ is positive, while $cos((b^k x_0 - \beta_k)\pi) \leq$

0. We can now use these inequalities to show that

$$\beta_k \geq \sum_{n=k+1}^{\infty} a^n \left( \frac{0}{\frac{\beta_k}{b^k} - x_0} \right) + a^k \left( \frac{1}{\frac{\beta_k}{b^k} + x_0} \right) + a^k \left( \frac{1}{\frac{\beta_k}{b^k} + x_0} \right) = \frac{(ab)^k}{\beta_k - b^k x_0} \tag{2.20}$$

And since $\beta_k - b^k x_0 < \frac{3}{2}$, then $(ab)^k \frac{1}{\beta_k - b^k x_0} > \frac{(ab)^k}{\frac{3}{2}} = \frac{2(ab)^k}{3}$, and therefore

$$(ab)^k \frac{2}{3} \leq |B_k| \tag{2.21}$$

Now that we've verified inequalities 2.8 and 2.9, we can use them to complete our contradiction. Recall from our initial conditions that $\frac{\pi}{ab-1} < \frac{2}{3}$ and $ab > 1$. Considering these conditions we can deduce that

$$\lim_{k\to\infty} (ab)^k (\frac{2}{3}) - (ab)^k (\frac{\pi}{ab-1}) = \lim_{k\to\infty} (ab)^k (\frac{2}{3} - \frac{\pi}{ab-1}) = \infty, \tag{2.22}$$

and so

$$\lim_{k\to\infty} (\frac{f(\frac{\beta_k}{b_k}) - f(x_0)}{\frac{\beta_k}{x-0}})(-1)^{\beta_k} = \infty \tag{2.23}$$

as well. Finally, since equation 2.23 approaches $\infty$, we know $f\prime(x_0)$ does not exist, and if the deriviative does not exist for any value of $x_0$, then $f(x)$ is not differentiable at any real number point. Therefore, we have proved by contradiction that the function $f$ is continuous everywhere, but differentiable nowhere. $\square$

Many generalizations have been built upon the Weierstrass function. One of the most famous includes Godfrey Hardy's proof that $\sum_{n=1}^{\infty} a^n sin(b^n x\pi)$ and $\sum_{n=1}^{\infty} a^n cos(b^n x\pi)$ are both continuous but differentiable nowhere on the real number system if $0 < a < 1$, $b > 1$, and $ab \geq 1$.

# 3. BOLZANO'S FUNCTION

"My special pleasure in mathematics rested particularly on its purely speculative part"
- Bernard Bolzano

While Weierstrass was the first to publicly present an everywhere continuous, nowhere differentiable function, he was not necessarily the first to discover such a function. Over four decades prior, mathematician and theologist; Bernard Bolzano, unbeknownst to himself, found a function with the aforementioned properties when trying to prove that his continuous function has isolated points which cannot be differentiated, and that these sets of points are everywhere dense and infinite within an interval $[a, b]$. In fact, Bolzano falsely believed that a continuous function must be differentiable except for some isolated values.

The function's non-differentiability wasn't brought to light until long after the death of Bernard Bolzano and was headed by a group of mathematicians in the early twentieth century. This group was known as the Bolzano Committee. It was specifically the discovery of Bolzano's function in his (at the time) unpublished piece, *Functionenlehre* by a teacher named Martin Jasek that sparked interest in the publication of Bolzano's manuscripts. While financial support and motivation of the group waned, with much of the unpublished work remaining unpublished and the original group disbanding in the 1950's, the committee still had the opportunity to present Bolzano's function to the public at the Bohemian Society of Sciences in December of 1921. In addition to this, photocopies of many of Bolzano's original manuscripts exist still at the Central Archives of the Academy of Sciences of Czech Republic [5]. Bolzano's function can be described as a limit of continuous functions on an interval $[a, b]$, such that the first continuous function; $y_1$ is linear on $[a, b]$, and also that

$$y_1(a) = A$$

$$y_1(b) = B$$

$$y_1(x) = A + (x - a)\frac{B - A}{b - a} \tag{3.1}$$

The next step of the function is determined by dividing $[a, b]$ into four subintervals, whose endpoints occur at $a$, $a + \frac{3}{8}(b - a)$, $\frac{1}{2}(a + b)$, $a + \frac{7}{8}(b - a)$, $b$.

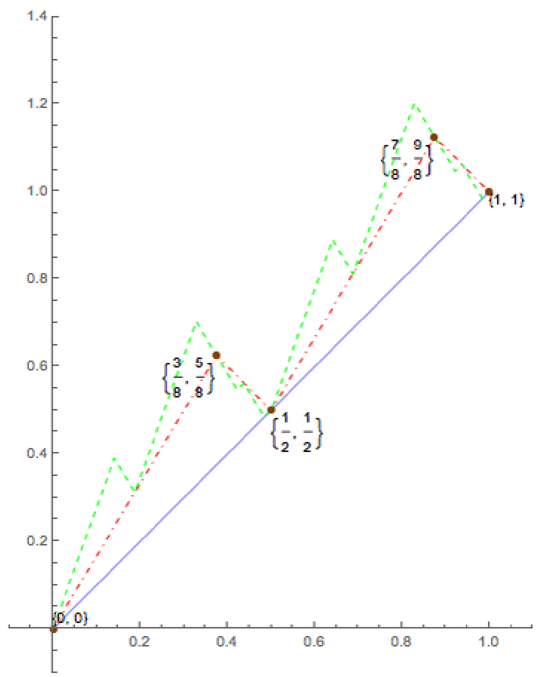


Figure 3.1: This graph comes from a Wolfram Alpha applet and depicts Bolzano's function's first three iterations in black, red, and green respectively. [13].

The following proof closely follows Johan Thim's strategy for showing nowhere-differentiability (which itself follows Bolzano's original results), although I have done all of my own calculations and simplified where I could [6]. I also chose to prove Bolzano's original result which is that the function is nowhere differentiable and continuous specifically on a dense subset of the interval $[a, b]$ rather than showing its nondifferentiability everywhere.

**Theorem 3.0.1.** *Bolzano's function is continuous everywhere, but differentiable nowhere on the dense subset of* $[a, b]$.

*Proof.* Using equation 3.1 which we defined prior, we can now define $y_2(x)$ on the following intervals:

$$I_1 = [a, a + \frac{3}{8}(b-a)],$$

$$I_2 = [a + \frac{3}{8}(b-a), \frac{1}{2}(a+b)],$$

$$I_3 = [\frac{1}{2}(a+b), a + \frac{7}{8}(b-a)]$$

$$I_4 = [a + \frac{7}{8}(b-a), b] \tag{3.2}$$

such that the endpoints of each piece of the piecewise linear function occur at the following values:

$$y_2(a) = A$$

$$y_2(b) = B$$

$$y_2(a + \frac{3}{8}(b-a)) = A + \frac{5}{8}(B-A)$$

$$y_2(\frac{1}{2}(a+b)) = A + \frac{1}{2}(B-A)$$

$$y_2(a + \frac{7}{8}(b-a)) = B + \frac{1}{8}(B-A) \tag{3.3}$$

The following functions : $y_3(x), y_4(x), ..., y_n(x)$ are constructed using the same mechanics as $y_2(x)$, with Bolzano's function; $y(x)$ being defined as

$$y(x) = \lim_{n\to\infty} y_n(x) \tag{3.4}$$

Now that we've established how iterations are built, to prove the function defined by equation 3.4 is continuous everywhere but nowhere differentiable, we'll follow the same format as the Weierstrass proof and begin by proving its continuity. To accomplish this, we'll be utilizing the following theorems:

**Theorem 3.0.2.** *The sequence $S_n$ converges uniformly if and only if $S_n$ is a uniformly Cauchy sequence on the interval I.*

**Theorem 3.0.3.** *The sequence $S_n$ is called a uniformly Cauchy sequence on the interval I if and only if* $\lim_{n,m\to\infty} sup|S_n(x) - S_m(x)| = 0$*, where* $x \in I$.

**Theorem 3.0.4.** *Let $S_n$ be a sequence of continuous functions on the intervals I and $S_n$ which uniformly converges to $S$ on I. It then follows that $S$ is also continuous on I.*

We'll begin by finding the slopes of the linear functions on their subintervals. Let $k$ be a variable within the natural number system. We can then use the defintion of the Bolzano function to find the slope of $y_k(x)$ for all $[a_k, b_k$ when $k = 1$

$$M_k = M_1 = \frac{B-A}{b-a} \tag{3.5}$$

Next we'll consider the case when $k > 1$. Recall the intervals $I_1, I_2, I_3, I_4$. For the interval $I_1$:

$$M_{k+1} = \frac{y_k(a_k + \frac{3}{8}(b_k - a_k - y_k(a_k)}{a_k + \frac{3}{8}(b_k - a_k) - a_k} = \frac{\frac{5}{8}(B_k - Ak)}{\frac{3}{8}(b_k - a - k} = (\frac{5}{3})(\frac{B_k - A_k}{b_k - ak} = \frac{5}{3}M_k \tag{3.6}$$

For $I_2$:

$$M_{k+1} = \frac{y_k(\frac{1}{2}(a_k + b_k)y_k(a_k + \frac{3}{8}(b_k - a_k)}{\frac{1}{2}(a_k + b_k) - a_k + \frac{3}{8}(b_k - a_k)} = \frac{(\frac{1}{2} - \frac{5}{8})(B_k - A_k)}{(\frac{1}{2} - \frac{3}{8})(b_k - a_k)} = -\frac{B_k - A_k}{b_k - a_k} = -M_k \tag{3.7}$$

For $I_3$

$$M_{k+1} = \frac{y_k(a_k + \frac{7}{8}(b_k - a_k)) - y_k(\frac{1}{2}(a_k + b_k)}{a_k + \frac{7}{8}(b_k - a_k) - \frac{1}{2}(a_k + b_k)} = \frac{(\frac{9}{8} - \frac{1}{2})(B_k - A_k)}{(\frac{7}{8} - \frac{1}{2})(b_k - a_k)} = \frac{\frac{5}{8}B_k - A_k}{\frac{3}{8}b_k - a_k} = \frac{5}{3}M_k \tag{3.8}$$

Finally, for $I_4$

$$M_{k+1} = \frac{y_k(b_k) - y_k(a_k + \frac{7}{8}(b_k - a_k))}{b_k - a_k + \frac{7}{8}(b_k - a_k)} = \frac{(\frac{-1}{8})(B_k - A_k)}{(1 - \frac{7}{8})(b_k - a_k} = -M_k \tag{3.9}$$

Next, we'll define the subintervals on $[a, b]$ for which Bolzano's function; $y_n$ is linear as $[I_n(s_k), I_n(t_k)]$. We can then use this to define the maximum length; $L$ an interval could have when $y_{n+1}$ is linear as

$$L_n = sup(I(t_k), I(s_k)). \tag{3.10}$$

With this in mind, we can also state that the maximum slope: $M_n$ of $y_{n+1}$ is the least upper bound of the slopes on the set of subintervals; $[I_n(s_k), I_n(t_k)]$ (which we'll equate to $I_{n,k}$ for clarity and brevity). That is,

$$M_n = sup(|M_n^i(I)|) \tag{3.11}$$

It is apparent then that

$$M_n \leq (\frac{5}{3})^{n+1}|\frac{B - A}{b - a}| \tag{3.12}$$

and

$$L_n \leq (\frac{3}{8})^{n+1}|b - a| \tag{3.13}$$

This is helpful because we now can state that the maximum amount of change between functions

$y_n$ and $y_{n+1}$ will be bounded by $M_n \cdot L_n$, and

$$M_n L_n \leq (\frac{5}{8})^{n+1}|B - A| \tag{3.14}$$

This also tells us that

$$(\frac{5}{8})^{k+1}|B - A| \geq sup_{x\in[a,b]}|y_{k+1}(x) - y_k(x)| \tag{3.15}$$

where $k \in \mathbb{N}$. Now let $n < m$ where $n, m \in \mathbb{N}$ so

$$sup_{x\in[a,b]}|y_m(x) - y_n(x)| \leq sup_{x\in[a,b]} \sum_{k=n+1}^{m} |y_k(x) - y_{k-1}(x)| \leq \sum_{k=n+1}^{m} (\frac{5}{8})^k|B - A| \tag{3.16}$$

Then since

$$\sum_{k=n+1}^{m} (\frac{5}{8})^k|B - A| = |B - A|(\sum_{k=1}^{m}(\frac{5}{8})^k - \sum_{k=1}^{n}(\frac{5}{8})^k \tag{3.17}$$

we can state that

$$\lim_{m,n\to\infty} |B - A|(\frac{5}{8} - \frac{5}{8}) = 0. \tag{3.18}$$

Thus by theorem 3.03, we've proven $y_K$ is a uniformly Cauchy sequence on the interval $[a, b]$. Then, since we know any $y_k \in y_k$ is continuous, we know by theorems 3.02 and 3.04 that Bolzano's function is continuous on the interval $[a, b]$.

We now shift our attention to the differentiability of Bolzano's function. Again consider $[I_n(s_k), I_n(t_k)]$ to be the set of subintervals for which the function $y$ is linear and let $E$ be the set of endpoints within this set of intervals. Rather than prove nowhere differentiability, I'll be showing that $y$ isn't differentiable on a dense subset of the interval $[a, b]$.

Firstly, we must prove that $E$ is dense in the interval $[a, b]$ where $x_0$ is a fixed value within $[a, b]$. Assuming $x_0 \neq b$, we can show there exists an $i_0 \in 0, 1, 2, 3$ for which $x_0 \in K_0^{i_0}$ when we define $K_0^{i_0}$ as the following:

$$K_0^{(0)} = [a, a + \frac{3}{8}(b - a)) \tag{3.19}$$

$$K_0^{(1)} = [a + \frac{3}{8}(b - a), a + \frac{1}{2}(b - a)) \tag{3.20}$$

$$K_0^{(2)} = [a + \frac{1}{2}(b - a), a + \frac{7}{8}(b - a)) \tag{3.21}$$

$$K_0^{(3)} = [a + \frac{7}{8}(b - a), b) \tag{3.22}$$

We can then also claim $K_0^{(i_0)} = K_0$. Now consider that $x_0 \in [a_n, b_n]$, and that $[a_n, b_n] = I_{n-1}$. We now have

$$K_n^{(0)} = [a_n, a_n + \frac{3}{8}(b_n - a_n)) \tag{3.23}$$

$$K_n^{(1)} = [a_n + \frac{3}{8}(b_n - a_n), a_n + \frac{1}{2}(b_n - a_n)) \tag{3.24}$$

$$K_n^{(2)} = [a_n + \frac{1}{2}(b_n - a_n), a + \frac{7}{8}(b_n - a_n)) \tag{3.25}$$

$$K_n^{(3)} = [a_n + \frac{7}{8}(b_n - a_n), b_n) \tag{3.26}$$

So similarly to the step before this, we can now claim $i_n \in 0, 1, 2, 3$ exists for which $x_0 \in K_n^{(i_0)}$, and $K_n^{(i_0)} = K_n$. It's also now apparent that

$$|x_0 - a_{n+1}| \leq (\frac{3}{8})^{n+1}|b - a| \tag{3.27}$$

and since

$$\lim_{n\to\infty}(\frac{3}{8})^{n+1} = 0 \tag{3.28}$$

we've proven that the set of endpoints; $E$, is dense in the interval $[a, b]$, since any point in $E$ is a limit of a point in $[a, b]$. We must next prove that for any value of $x_0$ within our set of endpoints, $y$ is nondifferentiable, or $y\prime(x_0)$ does not exist when we let $x_0 \in E$ be arbitrary but fixed.

So, let $x_0 \in E$ be arbitrary but fixed, and for our first scenario, consider that $x_0 \neq a$. We'll define $x_n$ as

$$x_n = x_0 - (\frac{1}{8})^{n+q}|b - a| \tag{3.29}$$

where $n, q \in \mathbb{N}$. There exists a natural number $r$ such that if $r \leq p$, then $y(x_0) = y_p(x_0)$

Next, choose a variable $q$ such that $r < q$, and $y(x_0) = y_n(x_0)$.

Keeping this and the structure of $y$ in mind, let there exist a real number $R$ such that

$$R \geq \frac{|b - a|}{|B - A|} \neq 0 \tag{3.30}$$

We can use these variables to say

$$y(x_n) = y_{n+1}(x_n) = y_n(x_0) + (-1)^n(\frac{1}{8})^{n+q}(R) \tag{3.31}$$

and then that

$$[y(x_0 - y_n(x_0)](8)^{n+q} = [y_n(x_0 - y_n(x_0)](8)^{n+q} = 0 \tag{3.32}$$

To continue,

$$\frac{y(x_0) - y(x_n)}{x_0 - x_n} = (8)^{n+q}[y(x_0) - y_n(x_0) - (-1)^n(\frac{1}{8})^{n+q}(R)] \tag{3.33}$$

$$= (y(x_0) - y_n x_0)(8)^{n+q} - (-1)^n(R) \tag{3.34}$$

$$= (-1)^n(R) = (-1)^{n+1}(R) \tag{3.35}$$

It's evident that as $n$ approaches $\infty$, $(-1)^{n+1}(R)$ does not converge, and so $y'(x_0)$ does not exist.

We must now consider the case when $x_0 = a$. In this case, let $x_n$ be defined as

$$x_n = a + (\frac{3}{8})^n|b - a|. \tag{3.36}$$

It's apparent then that $n$ approaches $\infty$, $x_n$ approaches $a$, and therefore $x_n \in E$ for any natural number $n$. We can now see that

$$y(x_n) = y_{n+1}(x_n) \tag{3.37}$$

and

$$y(a) = A \tag{3.38}$$

and hence

$$y_{n+1}(x_n) = A + (\frac{5}{3})^n(\frac{3}{8})^n|b - a| \tag{3.39}$$

We can now prove $y\prime(x_0)$ does not exist similarly to how we did for the case in which $x_0 \in E\ a$. Consider that

$$\frac{y(x_n) - y(a)}{x_[n - a} = \frac{A + (\frac{5}{3})^n(\frac{3}{8})^n)|b - a| - A}{(\frac{3}{8})^n}|b - a| = (\frac{5}{3})^n. \tag{3.40}$$

It's obvious that as $n$ approaches $\infty$, $(\frac{5}{3})^n$ also approaches $\infty$ and therefore does not converge, meaning $y\prime(x_0)$ does not exist We have now proven that Bolzano's function; $y$, is non-differentiable on the dense subset of [a,b], Therefore, we have proven theorem 3.01.

□

# 4. PEANO AND HILBERT CURVES

> "Questions that pertain to the foundations of mathematics, although treated by many in recent times, still lack a satisfactory solution. Ambiguity of language is philosophy's main source of problems. That is why it is of the utmost importance to examine attentively the very words we use." - Giuseppe Peano

> "A mathematical theory is not to be considered complete until you have made it so clear you can explain it to the first man whom you meet on the street." - David Hilbert

Giuseppe Peano was an Italian mathematician born in the mid-nineteenth century, and spent the majority of his life teaching math at the University of Turin. Not unlike the way Weierstrass and Bolzano's discoveries were motivated by their need to solidify and standardize analysis and the definitions of continuity introduced by mathematicians like Cauchy, Peano also wished to formalize and improve upon the ideas of those before him. He is perhaps most well-known for his contributions to mathematical logic and formalizing arithmetic, specifically through his axioms for the natural number system [19].

In the context of the development of continuous, nowhere differentiable functions, it is Peano's continuous, nowhere differentiable curve introduced in 1890 that is his most significant contribution. This curve was also the first space-filling curve introduced and motivated the discovery of many variants. One of the most famous variants of Peano's function is Hilbert's curve [14].

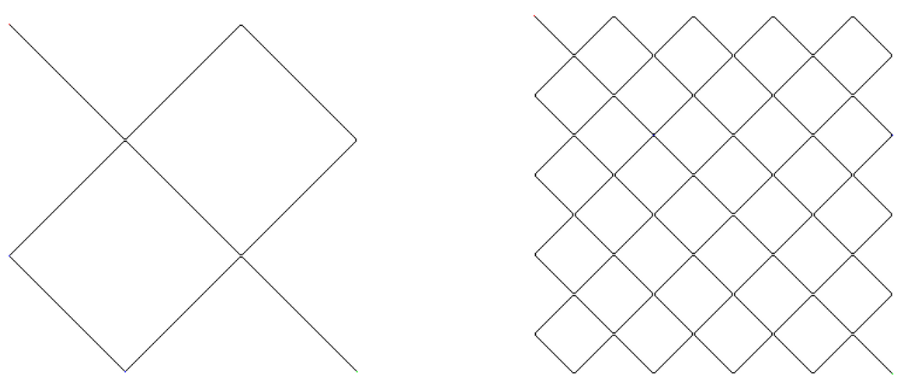

Figure 4.1: This figure depicts two iterations of Peano's function, and depicts the curve already beginning to fill the space of the unit square [10].

Just a year later, German mathematician David Hilbert introduced another continuous fractal, space-filling curve. Hilbert's curve maps the interval [0, 1] onto the unit square based on a ternary system. Let

$$t = \sum_{k=1}^{\infty} t_k 3^{-k} \tag{4.1}$$

where $t_k \in 0, 1, 2$ This is the ternary representation $t$ on the interval $[0, 1]$. Also, denote an operator $k$ as

$$kt_k = 2 - t_k. \tag{4.2}$$

The function maps the ternary fraction $(t_1, t_2, t_3, ...)$ on $[0, 1]$ to a point

$$x((t_1, t_2, t_3...)) = (t_1(k^{t_2}t_3)(k^{t_2+t_4}t_5)(k^{t_2+t_4+t_6}t_7)...),$$

$$y(t_1, t_2, t_3...) = ((k^{t_1}t_2)(k^{t_1+t_3}t_4)(k^{t_1+t_3+t_5}t_6)...) \tag{4.3}$$

in the unit square $[0, 1] \times [0, 1]$.

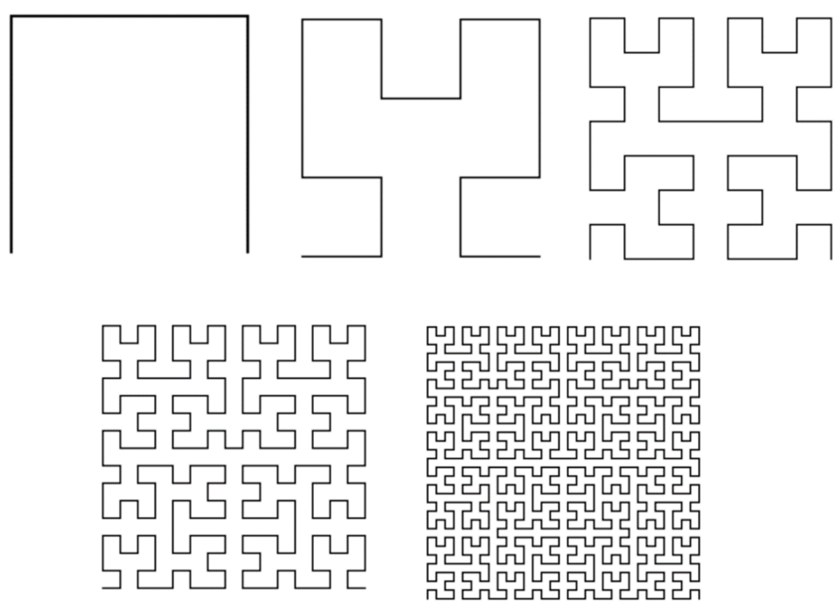

Figure 4.2: This figure depicts the first five iterations of the Hilbert curve we previously defined [9].

Hilbert proved the everywhere continuity and nowhere differentiability of his curve by defining two, single-valued component functions

$$x = \phi(t) \tag{4.4}$$

$$y = \psi(t) \tag{4.5}$$

of the Hilbert curve, and showing each component is continuous from either side of the interval [0, 1] before showing that

$$|\frac{\phi(t) - \phi(t_n)}{t - t_n}| = 3^n \tag{4.6}$$

which approaches $\infty$ as $n$ approaches $\infty$. This holds true for $\psi'(t)$ as well, proving the nowhere differentiability of the Hilbert curve on the interval [0, 1] [1][15]. The Hilbert curve is also a good example of a continuous, nowhere differentiable function with a significant real-world application, which will be explored further at the end of the paper.

# 5. KOCH'S CURVE

Helge Von Koch was a Swedish mathematician who both attended and was employed by schools in Stockholm for the majority of his life. While he wrote an abundance of papers, his work prior to the formation of the Koch curve and snowflake has been described by M. Berkopf in *The Dictionary of Scientific Biography* as "fairly accessible, although many of the calculations are lengthy" [18].

It was 1906 when Koch published *Une méthode géométrique élémentaire pour l'étude de certaines questions de la théorie des courbes plane* and introduced an infinitely long curve with a finite area, that is tangent nowhere. Not unlike how Weierstrass helped to dismantle the notion that all continuous functions must be differentiable except for at some isolated values, Koch wished to do the same but from a geometrical point of view that could be applied to curves [18].

Beginning with a straight-line segment, divide said segment into three equal parts, removing the middle part and making it instead one of two sides of an equilateral triangle, which replace the segment that is being removed. This construction results in us now having four line segments instead of one, for which each segment we can repeat the construction and end up with sixteen segments [7].

Koch's curve is one of the earliest mentioned examples of a fractal curve. Koch proved its continuity by first proving the curve is homeomorphic to $[0, 1]$ and can be parameterized into functions, say $f(t) = x$ and $g(t) = y$, which are both continuous and nowhere differentiable on $[0, 1]$ when $t \in [0, 1]$ [7].

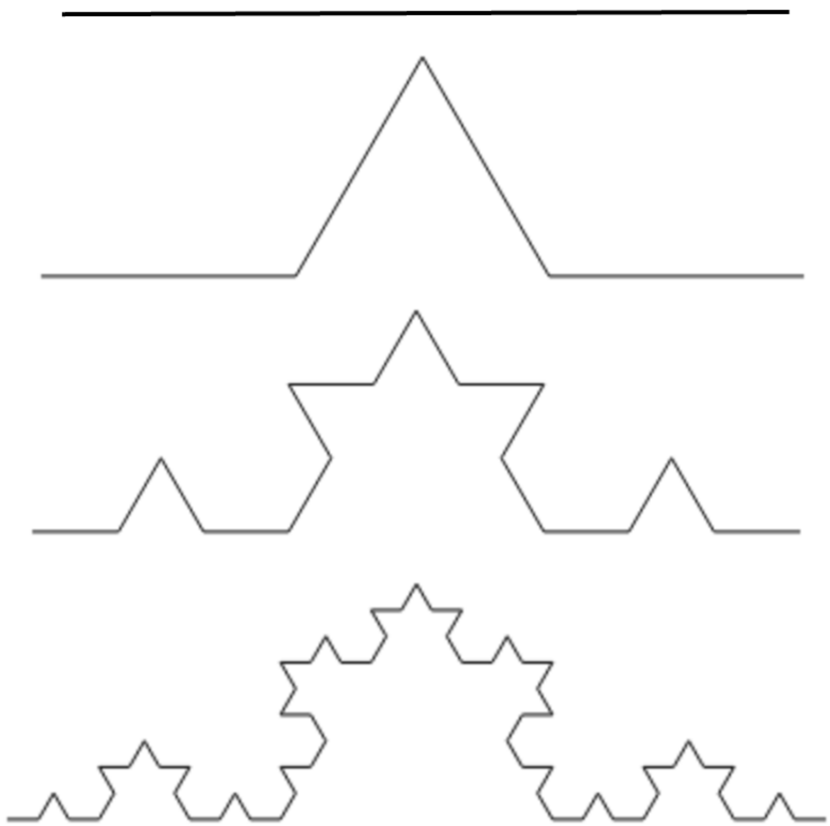

Figure 5.1: This figure depicts the first three iterations of Koch's curve. Koch's snowflake is constructed using the same process, but on each side of an equilateral triangle [11].

# 6. LIU WEN'S FUNCTION

Liu Wen's everywhere continuous, nowhere differentiable function is one of the more recent discoveries of such a function, having been published in 2002 [8]. It is also unique in the fact that it is constructed by infinite product and can be defined as

$$W(x) = \prod_{n=1}^{\infty}(1 + a_n \sin(b_n \pi x)) \tag{6.1}$$

where $W : \mathbb{R} \to \mathbb{R}$ and $0 < a_n < 1$. Also, $\sum_{k=1}^{\infty} a_k < \infty$ and $b_n = \prod_{k=1}^{n} p_k$ where $n, k \in \mathbb{N}$ and $p_k$ is even. Also, $\lim_{n\to\infty} \frac{2^n}{a_n p_n} = 0$

Now that we've defined the function, we'll shift our focus to proving its continuity and nondifferentiability. My proof follows Liu Wen's original piece published in 2002, but is more specific and includes steps which Wen may have left to the readers intuition [8].

**Theorem 6.0.1.** *The function $W(x)$ is continuous everywhere, but differentiable nowhere.*

*Proof.* Similar to the two proofs preceding this, we'll begin by showing the $W(x)$ we defined is continuous, keeping in mind the following useful theorem and inequality:

**Theorem 6.0.2.** *If $f_n : I \to \mathbb{R}$ is continuous for any $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(f_n(x))$ converges uniformly to $S(x)$ on the interval I, then $S$ is also a continuous function on I.*

$$\frac{x}{1+x} \leq \ln(1+x) \leq x \tag{6.2}$$

when $x \geq 0$.

Let $a$ represent the maximum possible value of $a_n$ when $n$ is a positive integer. Considering that we know $0 < a_n < 1$, we can extend this to include $0 < a < 1$. Then by inequality 6.2, we can show that

$$\begin{aligned}
&|\ln(1 + a_n \sin(b_n \pi x))| \\
&\leq a_n |\sin(b_n \pi x)| \cdot \max[\frac{1}{|1 + a_n \sin(b_n \pi x)|}, 1] \\
&\leq a_n \cdot \max[\frac{1}{1-a}, 1] \leq \frac{a_n}{1-a}
\end{aligned} \tag{6.3}$$

Then, since $\sum_{k=1}^{\infty} a_k < \infty$, we can apply theorem 2.02; Weierstrass's M-test to state that $\sum_{k=1}^{\infty} \ln((1 + a_n sin(b_n \pi x))$ uniformly converges to a continuous function, and therefore through theorem 6.02, we can also state that

$$W(x) = \prod_{n=1}^{\infty}(1 + a_n sin(b_n \pi x)) = \exp(\sum_{k=1}^{\infty} \ln((1 + a_n sin(b_n \pi x))) \tag{6.4}$$

is continuous.

Next, we need to prove $W(x)$ is nowhere differentiable. Consider that for any $x \in \mathbb{R}$ there exists a sequence of integers $S_n$ with $n \in \mathbb{N}$ for which $x \in [\frac{S_n}{b_n}, \frac{S_n+1}{b_n})$.

Now we'll define the following sequences as

$$y_n = \frac{S_n + 1}{b_n} \tag{6.5}$$

$$z_n = \frac{S_n + \frac{3}{2}}{b_n} \tag{6.6}$$

so that it is obvious that

$$x < y_n < z_n \tag{6.7}$$

Also,

$$0 < z_n - x < \frac{3}{2b_n} \tag{6.8}$$

and

$$z_n - y_n = \frac{1}{2b_n} \tag{6.9}$$

and

$$z_n - x = \frac{S_n + \frac{3}{2}}{b_n} - x \leq \frac{S_n + \frac{3}{2}}{b_n} - \frac{S_n}{b_n} = \frac{3}{2b_n} \tag{6.10}$$

Then since we've shown $z_n - y_n \leq \frac{3}{2b_n}$ and $\frac{3}{2b_n} = 3(z_n - yn)$, we can form the following inequality:

$$z_n - y_n \geq \frac{1}{3}(z_n - x) > \frac{1}{3}(y_n - x). \tag{6.11}$$

Now for clarity, define $a$ and $b$ as

$$a = \prod_{k=1}^{\infty}(1 - a_k) \tag{6.12}$$

$$b = \prod_{k=1}^{\infty}(1 + a_k) \tag{6.13}$$

and define the functions $L_n(x)$ and $\delta_n$ as

$$L_n(x) = \prod_{k=1}^{n}(1 + a_k \sin(b_k \pi x)) \tag{6.14}$$

$$\delta_n = \prod_{k=1}^{\infty}(1 + a_k \sin(b_k \pi z_n)) - \prod_{k=1}^{\infty}(1 + a_k \sin(b_k \pi y_n)) \tag{6.15}$$

where $L_n : \mathbb{R} \to \mathbb{R}$.

So if $k = n$, then

$$\sin(b_n \pi y_n) = \sin((S_n + 1)\pi) = 0 \tag{6.16}$$

and

$$\sin(b_n \pi z_n) = -(-1)^{S_n} \tag{6.17}$$

And if $k > n$, then

$$\sin(b_k \pi y_n) = \sin(2q_k(S_n + 1)\pi) = 0 \tag{6.18}$$

and

$$\sin(b_k \pi z_n) = \sin(3q_k \pi) = 0 \tag{6.19}$$

where $q_k$ is an arbitrary integer. Lastly, if $k < n$, then

$$|a_k \sin(b_k \pi z_n) - a_k \sin(b_k \pi y_n)| \leq a_k |b_k \pi (z_n - y_n)| < \frac{\pi}{2p_n}. \tag{6.20}$$

Considering this, we can redefine $\delta_n$ as

$$\delta_n = L_{n-1}(z_n) - L_{n-1}(y_n) - (-1)^{S_n}(a_n)(L_{n-1}(z_n)). \tag{6.21}$$

Next, let there exist a real number $t_k$ such that $|t_k| < \frac{\pi}{2p_n} < 1$, and so

$$a_k \sin(b_k \pi z_n) = a_k \sin(b_k \pi y_n) + t_k. \tag{6.22}$$

To finish the proof, we'll find the bounds of $\delta_n$ and estimate its value. First we'll find the lower bound.

Consider that

$$|L_{n-1}(z_n) - L_{n-1}(y_n)| = |\prod_{k=1}^{n-1}((1 + a_k \sin(b_k\pi z_n)) + t_k) - \prod_{k=1}^{n-1}(1 + a_k \sin(b_k\pi y_n))|$$

$$\leq \sum_{i=1}^{2(n-1)-1} |t_{l_i}|(\prod_{j\in I_i}|t_k|)(\prod_{j\in J_i}|1 + a_j sin(b_j\pi y_n)|)$$

$$\leq (\frac{\pi}{2p_n})(\sum_{i=1}^{2(n-1)-1}(\prod_{j\in J_i}|1 + a_j|)) \leq (\frac{b\pi}{2p_n})(2^{n-2}) \tag{6.23}$$

where $i \in I_i$ and both $I_i$ and $J_i$ are some index sets which are subsets of the natural number system, and $l_i$ is some index. To continue, we can again describe $\delta_n$ with the following inequality:

$$|\delta_n| = |(L_{n-1}(z_n) - L_{n-1}(y_n) - 1)S_n(a_n)(L_{n-1}(z_n))| \tag{6.24}$$

$$\geq a_n L_{n-1}(z_n) - |L_{n-1}(z_n) - L_{n-1}(y_n)|$$

$$\geq a_n a - \frac{b\pi}{p_n}(2^{n-2}) = a_n(a - \frac{2^{n-2}}{a_n p_n}b\pi)$$

since $a < L_n(x) < b$. Also, assume that $\lim_{n\to\infty} \frac{2^n}{a_n p_n} = 0$. Then, since $a_n b_n \geq a_n p_n$, it follows that $a_n b_n$ also approaches $\infty$ as n approaches $\infty$.

So we can show that

$$\lim_{n\to\infty} |\frac{W(z_n) - W(y_n)}{z_n - y_n}| = \lim_{n\to\infty} |2b_n\delta_n| \geq \lim_{n\to\infty} |2a_n b_n(a - \frac{2^{n-2}}{a_n p_n}b\pi)| = \infty \tag{6.25}$$

Then, since $z_n - y_n \geq \frac{1}{3}(y_n - x)$ and by the triangle inequality, we can state the following inequalities

$$|\frac{W(z_n) - W(y_n)}{z_n - y_n}| \leq \frac{|W(z_n) - W(x)|}{z_n - y_n} + \frac{|W(y_n) - W(x)|}{z_n - y_n} \leq \frac{3|W(z_n) - W(x)|}{z_n - x} + \frac{3|W(y_n) - W(x)|}{y_n - x} \tag{6.26}$$

The above inequalities show that $W(x)$ has no right derivative, and since $x$ is some arbitrary real value, it is implied that $W$ is nowhere differentiable from the right.

We could repeat a very similar process to this to show that $W$ also has no finite derivative at $x$ from the left either, proving that $W$ is nowhere differentiable. Therefore, we've showed that $W(x)$ is both continuous everywhere, but differentiable nowhere. □

# 7. CONCLUSION

Considering that Weierstrass's presentation of a continuous function with no well-defined tangent at any point was initially shunned by much of the mathematical society, particularly for its departure from discoveries being rooted in practical processes, it may be surprising that functions with these properties have a variety of real-world applications.

Firstly, the Weierstrass function, as well as the Koch, Peano, and Hilbert curves are all examples of fractals, despite the term "fractal" not being coined until 1975 [14]. In fact, Weierstrass's function is the first known graph of a fractal curve. Fractals are unique in the fact that they can be used to described shapes which cannot be defined by traditional, Euclidean geometry. For instance, shore lines, clouds, and many other organic structures closely resemble and can be mimicked by fractals. Even the growth pattern of bacteria can be represented and predicted using fractal geometry [16].

Everywhere continuous, nowhere differentiable curves are also used in computer graphics, For instance, fractals have been used in television shows like *Star Trek* to build landscapes and skies which look realistic. The Hilbert curve specifically plays an important role in computer science and image compression. An example of this is using the Hilbert curve to map the range of IP addresses a computer may use from a 2D to a 1D image . In fact, Google uses this curve for cache locality, as it keeps entries which are similar in value close together spatially post-mapping. In the case of IP addresses, nearby IP addresses will also be near in the processed image [17].

This paper discusses just a small handful of significant discoveries of continuous, nowhere differentiable functions. Not only do a wide array of perhaps more notable historical examples exist; like Takagi and van der Waerden functions, the Sierpiński curve, or the Knopp function to name a few, but new functions with these properties are still being discovered. Liu Wen has published multiple generalizations of his own function since proving the continuity and nowhere differentiability of his original infinite product [6].

While the functions are mathematically interesting since they defy one's intuition, the reaction to their publication also gives a fascinating insight into how academic communities have reacted to pathological discoveries throughout the past couple centuries.

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